

Interacting Stochastic Process and Renormalization Theory

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Abstract

A stochastic process with self-interaction as a model of quantum field theory is studied. We consider an Ornstein-Uhlenbeck stochastic process $x(t)$ with interaction of the form $x^{(\alpha)}(t)^4$, where α indicates the fractional derivative. Using Bogoliubov's R -operation we investigate ultraviolet divergencies for the various parameters α . Ultraviolet properties of this one-dimensional model in the case $\alpha = 3/4$ are similar to those in the φ_4^4 theory but there are extra counterterms. It is shown that the model is two-loops renormalizable. For $5/8 \leq \alpha < 3/4$ the model has a finite number of divergent Feynman diagrams. In the case $\alpha = 2/3$ the model is similar to the φ_3^4 theory. If $0 \leq \alpha < 5/8$ then the model does not have ultraviolet divergencies at all. Finally if $\alpha > 3/4$ then the model is nonrenormalizable.

This model can be used for a non-perturbative study of ultraviolet divergencies in quantum field theory and also in theory of phase transitions.

1 Introduction

There is a very fruitful interrelation between probability theory and quantum field theory [1]-[9]. In this note we consider a stochastic process that shows

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the same divergencies as quantum electrodynamics or φ^4 theory in the 4-dimensional spacetime. Therefore this one-dimensional model can be used for studying the fundamental problem of non perturbative investigation of renormalized quantum field theory [1, 3]. It can also find applications in theory of phase transitions [6, 7, 8].

The simplest nontrivial model of quantum field theory in d -dimensional spacetime is the model of scalar field with the φ^4 -interaction. In one dimensional case ($d = 1$) this model is equivalent to an anharmonic oscillator and it does not have ultraviolet divergencies. In this note we will study a more interesting model with the interaction that contains fractional derivatives. It is well known that trajectories of the Wiener process are Hölder-continuous with exponent $\alpha < 1/2$. The property of a function to be Hölder-continuous is related to the property of having a fractional derivative.

Let us remind that the free scalar *massless* quantum field on the semi-axis is the Wiener process. The scalar *massive* one-dimensional quantum field is an Ornstein-Uhlenbeck stochastic process $x(t) = x(t, \omega)$ [2]. In order to introduce an *interacting* Ornstein-Uhlenbeck stochastic process we will use fractional derivatives $x^{(\alpha)}(t)$ ($0 \leq \alpha < 1$) [10, 11]. Stochastic differential equations with fractional derivatives [12] are considered in [13] on p -adic number fields.

In this note an interacting Ornstein-Uhlenbeck stochastic process with interaction of the form $x^{(\alpha)}(t)^4$ will be discussed. Using Bogoliubov's R -operation we investigate ultraviolet divergencies for various parameters α . Ultraviolet properties of this one-dimensional model in the case $\alpha = 3/4$ are similar to those in the φ_4^4 theory but there are extra counterterms. It is shown that the model is two-loops renormalizable. For $5/8 \leq \alpha < 3/4$ the model has a finite number of divergent Feynman diagrams. In the case $\alpha = 2/3$ the model is similar to the φ_3^4 theory. If $0 \leq \alpha < 5/8$ then the model does not have ultraviolet divergencies at all. Finally if $\alpha > 3/4$ then the model is nonrenormalizable.

This paper is organized as follows. In the next section the interacting stochastic process is defined. Feynman diagrams for this process are described in Sect.2. A brief discussion of R -operation is given in Sect.4. Analysis of divergent Feynman diagrams for the interacting stochastic process is presented in Sect.5. Finally, in Sect.6 interesting particular cases of the interacting stochastic process are discussed.

2 The Interacting Stochastic Process

Let $x(t) = x(t, \omega)$ be an Ornstein-Uhlenbeck stochastic process with the correlation function

$$E(x(t)x(\tau)) = F(t - \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ip(t-\tau)}}{p^2 + m^2} dp = \frac{e^{-m|t-\tau|}}{2m} \quad (1)$$

where $m > 0$. There exists a spectral representation of the Ornstein-Uhlenbeck stochastic process [14]

$$x(t, \omega) = \int e^{ikt} \zeta(dk, \omega)$$

where $\zeta(dk, \omega)$ is a stochastic measure. We define the fractional derivative as

$$x^{(\alpha)}(t, \omega) = \int |k|^\alpha e^{ikt} \zeta(dk, \omega) \quad (2)$$

If $0 \leq \alpha < 1/2$ then $x^{(\alpha)}(t)$ is a stochastic process. If $\alpha \geq 1/2$ then one needs a regularization described below. We will use distribution notations and write

$$\begin{aligned} \zeta(dk, \omega) &= \tilde{x}(k, \omega) dk \\ \tilde{x}(k, \omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t, \omega) e^{-ikt} dt \end{aligned}$$

We want to give a meaning to the following correlation functions

$$K(t_1, \dots, t_N) = E(x(t_1) \cdots x(t_N) e^{-\lambda U}) / E(e^{-\lambda U}) \quad (3)$$

for all $N = 1, 2, \dots$. Here

$$U = \int_{-\infty}^{\infty} x^{(\alpha)}(\tau)^4 g(\tau) d\tau \quad (4)$$

where $g(\tau)$ is a nonnegative test function with a compact support (the volume cut-off), $x^{(\alpha)}(t)$ denotes the fractional derivative (2) and $\lambda \geq 0$. We will denote the expectation value as $E(A) = \langle A \rangle$. In this notations

$$\langle x(t)x(\tau) \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ip(t-\tau)}}{p^2 + m^2} dp$$

If $\alpha \geq 5/8$ then the expectation value in (3) has no meaning even if we expand it into the perturbation series in λ because there are ultraviolet divergencies (see below). We have to introduce a cutoff stochastic process $x_{\varkappa}(t)$ [3]

$$x_{\varkappa}(t, \omega) = \int_{-\varkappa}^{\varkappa} e^{ikt} \zeta(dk, \omega)$$

Instead of U in (3) we put

$$U_{\varkappa} = \int : x_{\varkappa}^{(\alpha)}(\tau)^4 : g(\tau) d\tau$$

where

$$x_{\varkappa}^{(\alpha)}(t, \omega) = \int_{-\varkappa}^{\varkappa} |p|^{\alpha} \tilde{x}(p, \omega) e^{ipt} dp$$

Here

$$\begin{aligned} : x_{\varkappa}^{(\alpha)}(\tau)^4 : &:= \int_{-\varkappa}^{\varkappa} \cdots \int_{-\varkappa}^{\varkappa} dp_1 \cdots dp_4 |p_1|^{\alpha} \cdots |p_4|^{\alpha} e^{i\tau(\sum_{j=1}^4 p_j)} \\ &: \tilde{x}(p_1) \tilde{x}(p_2) \tilde{x}(p_3) \tilde{x}(p_4) : \end{aligned}$$

where the normal product is defined by the relation (Wick's theorem)

$$\begin{aligned} : \tilde{x}(p_1) \tilde{x}(p_2) \tilde{x}(p_3) \tilde{x}(p_4) : &:= \tilde{x}(p_1) \tilde{x}(p_2) \tilde{x}(p_3) \tilde{x}(p_4) \\ &- \langle \tilde{x}(p_1) \tilde{x}(p_2) \rangle : \tilde{x}(p_3) \tilde{x}(p_4) : + \cdots \\ &- \langle \tilde{x}(p_1) \tilde{x}(p_2) \rangle \langle \tilde{x}(p_3) \tilde{x}(p_4) \rangle - \cdots - \langle \tilde{x}(p_1) \tilde{x}(p_4) \rangle \langle \tilde{x}(p_2) \tilde{x}(p_3) \rangle \end{aligned} \quad (5)$$

and

$$\langle \tilde{x}(p) \tilde{x}(k) \rangle = \frac{1}{2\pi} \cdot \frac{\delta(p+k)}{p^2 + m^2}$$

The problem is to prove that after the renormalization there exists a limit of the correlation functions

$$\langle x(t_1) \cdots x(t_N) e^{-\lambda U_{\varkappa}} \rangle_{ren}$$

as $\varkappa \rightarrow \infty$. We will consider this problem below by using the Bogoliubov-Parasiuk R -operation.

Remark. Quadratic interaction

$$U = \int x^{(\alpha)}(\tau)^2 g(\tau) d\tau$$

is a well defined stochastic variable if $\alpha < 1/2$. One can go beyond the boundary $\alpha = 1/2$ if one takes the normal product

$$U = \int : x^{(\alpha)}(\tau)^2 : g(\tau) d\tau$$

An heuristic definition of the correlation function (3) is given by the functional integral

$$K(t_1, \dots, t_N) = \int \varphi(t_1) \cdots \varphi(t_N) e^{-S} \mathcal{D}\varphi / \int e^{-S} \mathcal{D}\varphi \quad (6)$$

where the action

$$S = \int \left[\frac{1}{2} \dot{\varphi}(t)^2 + \frac{m^2}{2} \varphi(t)^2 + \lambda g(t) \varphi^{(\alpha)}(t)^4 \right] dt \quad (7)$$

and $\varphi(t)$ is a real valued function on the real axis.

3 Feynman Diagrams

Feynman diagrams (or graphs) are convenient tools for labeling and recording terms that appear while integrating polynomials with respect to a Gaussian measure, see [1, 3]. For the Ornstein-Uhlenbeck stochastic process one has

$$\langle x(t_1) \dots x(t_l) \rangle = \sum_{\text{pairings}} F(t_{i_1} - t_{i_2}) \cdots F(t_{i_{l-1}} - t_{i_l}) \quad (8)$$

where $F(t)$ is defined by (1) and the sum extends over all distinct ways of choosing the pairs $\{t_{i_k}, t_{i_{k+1}}\}$.

For the correlation function (3) one has the perturbative expansion

$$\langle x(t_1) \dots x(t_N) e^{-\lambda U} \rangle = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \langle x(t_1) \dots x(t_N) U^n \rangle$$

Then by using (8) one writes

$$\begin{aligned} & \langle x(t_1) \cdots x(t_N) U^n \rangle = \\ & = \int d\tau_1 \cdots d\tau_N \langle x(t_1) \cdots x(t_N) x^{(\alpha)}(\tau_1)^4 \cdots x^{(\alpha)}(\tau_n)^4 \rangle = \end{aligned}$$

$$= \sum_{\{\Gamma\}} K_{\Gamma}(t_1, \dots, t_N)$$

where Γ ranges over a set of graphs and $K_{\Gamma}(t_1, \dots, t_N)$ is a function of t_1, \dots, t_N assigned to Γ . A graph (or diagram) is a collection of vertices (represented as points), lines (represented as line segments joining vertices), and legs (represented as line segments which attach at one endpoint only to a vertex).

First let us describe a Feynman graph that corresponds to

$$A = x(t_1) \dots x(t_N) x^{(\alpha)}(\tau_1)^4 \dots x^{(\alpha)}(\tau_n)^4$$

Each factor $x(t_i)$ is represented by a vertex with a leg and each factor $x^{(\alpha)}(\tau_i)^4$ in A is represented by a single vertex with 4 legs. Now the integral

$$\int d\tau_1 \dots d\tau_N \langle A \rangle$$

is represented by a sum of graphs obtained by pairing the legs in the graph corresponding to A in all possible manners. Examples of Feynman diagrams are presented in (Fig.1-4).

Now each variable t_j is represented by a leg. Each variable τ_i is represented by a vertex with 4 legs or lines. Each line joining vertices s_k and s_l (here s_k stands for t_j or τ_i) gives rise to a function $F(s_i - s_k)$ defined in (1). One has

$$K_{\Gamma}(t_1, \dots, t_N) = \int d\tau_1 \dots d\tau_N \prod_{\substack{\text{lines} \\ \text{legs}}} F(s_k - s_l)$$

In momentum representation we obtain the expression of the form

$$\langle \tilde{x}(p_1) \dots \tilde{x}(p_N) e^{-\lambda U} \rangle = \sum_{\{\Gamma\}} G_{\Gamma}(p_1, \dots, p_N)$$

The sum runs over all diagrams Γ with N external legs that can be build up using 4-vertices corresponding to the $x^{(\alpha)^4}$ term. Contributions from the connected diagrams with n 4-vertices come from the expression

$$\int \langle \tilde{x}(p_1) \dots \tilde{x}(p_N) (\prod_{i=1}^4 \tilde{x}(k_i^{(1)}) |k_i^{(1)}|^\alpha) \dots (\prod_{i=1}^4 \tilde{x}(k_i^{(n)}) |k_i^{(n)}|^\alpha) \rangle$$

$$\delta(\sum_{i=1}^4 k_i^{(1)}) \dots \delta(\sum_{i=1}^4 k_i^{(n)}) \prod_{i,j} dk_i^{(j)}$$

Using the pairing (5) we obtain the contribution from a diagram Γ with L internal lines in the form

$$\int (\prod_{l=1}^L \frac{|k_l|^{2\alpha}}{k_l^2 + m^2}) (\prod_{i=1}^n \delta(\xi_i)) dk_1 \dots dk_L \quad (9)$$

Here ξ_i denotes the linear combination of momenta assigned to the i -th vertex. We integrate over $(n-1)$ momenta using $\delta(\xi_i)$ and obtain

$$\delta(\sum_{j=1}^N p_j) \int I_\Gamma dk_1 \dots dk_l \quad (10)$$

where $l = L - (n-1)$. Here I_Γ is the unrenormalized integrand of the form

$$I_\Gamma = \prod_{j=1}^L \frac{|q_j|^{2\alpha}}{q_j^2 + m^2} \quad (11)$$

where q_l are linear combinations of the internal momenta k_1, \dots, k_L and external momenta p_1, \dots, p_N .

Remark. If we make a change of variables $x^{(\alpha)}(t) = y(t)$ in the original Lagrangian then we obtain an interacting stochastic process with $\lambda y(t)^4$ interaction and propagator

$$\langle \tilde{y}(p) \tilde{y}(k) \rangle = \frac{1}{2\pi} \cdot \frac{|p|^{2\alpha} \delta(p+k)}{p^2 + m^2}$$

This remark explains the appearance of the factor $|q_j|^{2\alpha}$ in (11).

4 R -Operation

A diagram Γ is called proper (or one-particle-irreducible) if it is connected and can not be separated in two parts by cutting a single line. The canonical

degree $D(\Gamma)$ of a proper diagram is defined by the dimension of the corresponding Feynman integral with respect to the integration variables. From (9) and (10) we have

$$D = D(\Gamma) = (2\alpha - 2)L + l = (2\alpha - 1)L - n + 1 \quad (12)$$

A diagram Γ is called superficially divergent if its dimension $D(\Gamma) \geq 0$. A proper diagram Γ which is superficially divergent ($D(\Gamma) \geq 0$) is called renormalization part. A Γ -forest W is a set of diagrams satisfying the following conditions:

- i) the elements of W are renormalization parts of Γ
- ii) any two elements of W are non-overlapping

Let I_Γ be an unrenormalized integrand. The Bogoliubov-Parasiuk prescription for the renormalized integral can be written as [1, 15]

$$R_\Gamma = \sum_W \prod_{\gamma \in W} (-t^\gamma) I_\Gamma$$

where the sum runs over all Γ -forests. Here $t^\gamma I_\Gamma$ denotes the Taylor series with respect to the appropriate external variables around zero up to order $D(\gamma)$. The renormalized integral R_Γ is considered as a result of action of R -operation to I_Γ . This formula is equivalent to the original recursive R -operation defined by using the reduced diagrams. It is one of the main results of the Bogoliubov-Parasiuk-Hepp-Zimmerman (BPHZ) theory that one has the absolute convergence of the renormalized integral

$$J_\Gamma = \int dk_1 \cdots dk_l R_\Gamma$$

The same result can be obtained if we introduce the cut-off \varkappa , then make a reparametrization of the parameters in the original (bare) Lagrangian, and finally remove the cut-off. For example the renormalized Euclidean interaction Lagrangian of the $\lambda\varphi^4$ theory in 4-dimensional space is [1]

$$\mathcal{L} = \frac{Z_3 - 1}{2} \left((\nabla\varphi)^2 + \frac{m^2}{2} \varphi^2 \right) + \frac{\delta m^2}{2} \varphi^2 + \lambda(Z_4 - 1)\varphi^4$$

Here Z_3 , Z_4 , and δm^2 are represented as series in λ and the corresponding coefficients in the expansions of Z_3 and Z_4 have logarithmic divergencies as $\varkappa \rightarrow \infty$ and coefficients of δm^2 have quadratic divergencies.

We will show at the two-loop level that in the $x^{(\alpha)^4}$ model with $\alpha = 3/4$ the renormalized Lagrangian has a similar structure

$$\begin{aligned}\mathcal{L} = & \frac{Z_3 - 1}{2} \left(\dot{x}(t)^2 + \frac{m^2}{2} x(t)^2 \right) + \frac{\delta m^2}{2} x(t)^2 \\ & + \frac{Z_2 - 1}{2} x^{(3/4)}(t)^2 + \lambda(Z_4 - 1) x^{(\alpha)}(t)^4\end{aligned}$$

Note the term $x^{(3/4)}(t)^2$ which is discussed below.

5 Divergent Diagrams

The action (7) and the correlation functions (3), (6) describe a one dimensional model of quantum field theory. Although one-dimensional it is an interesting and instructive model because it demonstrates the most striking property of quantum field theory, i.e. its ultraviolet divergencies. By expanding (3) or (6) into the perturbation series in λ one obtains the Feynman diagrams (9). Let us investigate the convergence of the analytical expressions corresponding to the Feynman diagrams by using the canonical degree $D(\Gamma)$ of the diagram (12)

$$D = L(2\alpha - 1) - n + 1 \quad (13)$$

If for a given diagram $D < 0$ then it is superficially finite, otherwise it is divergent.

Let us consider a proper diagram with n vertices, L internal lines, and E legs. We have the following relation

$$4n = 2L + E \quad (14)$$

Note that for any nontrivial connected diagram

$$2n \geq L \geq n \geq 2 \quad (15)$$

$$E \leq 2n \quad (16)$$

Theorem *If $\alpha < 5/8$ then all Feynman diagrams of the interacting stochastic process are superficially finite. If $5/8 \leq \alpha < 3/4$ then there exists a finite number of divergent diagrams, moreover all divergent diagrams have only 0 or 2 legs. If $\alpha = 3/4$ then the model is renormalizable and all divergent*

diagrams have only 0, 2 or 4 external lines. Finally, if $\alpha > 3/4$ then the model is nonrenormalizable.

Proof Let us prove the first statement of the theorem, i.e. if $\alpha < 5/8$ then $D < 0$ for any $n \geq 2$. Using (13) and (15) we have

$$\begin{aligned} D \Big|_{\alpha < 5/8} &< 2L \cdot \frac{5}{8} - L - n + 1 = \frac{L - 4n + 4}{4} \leq \\ &\leq \frac{2n - 4n + 4}{4} = \frac{2 - n}{2} \leq 0 \end{aligned} \quad (17)$$

From (17) it follows that $D < 0$ for any $\alpha < 5/8$.

Let us consider $\alpha = 5/8$. Similarly to (17) from (13) we have

$$D \Big|_{\alpha = 5/8} = \frac{L - 4n + 4}{4} \leq \frac{2 - n}{2} \leq 0 \quad (18)$$

Therefore only two-point ($n = 2$) diagram could be divergent (in this case $D = 0$). Rewriting (18) in the form

$$D \Big|_{\alpha < 5/8} = \frac{4 - (E + L)}{4} \quad (19)$$

From (19) it follows that only diagram with $E = 0$, $L = 4$, $n = 2$ (Fig. 3) is divergent.

In the case when $5/8 < \alpha < 3/4$ we can write

$$\alpha = \frac{3}{4} - \varepsilon \quad (20)$$

where $0 < \varepsilon < 1/8$. Substituting (20) into (13) and using (15) we have

$$D \Big|_{\alpha = 3/4 - \varepsilon} = \frac{L}{2} - 2L\varepsilon - n + 1 \leq \frac{2n}{2} - 2n\varepsilon - n + 1 = 1 - 2n\varepsilon \quad (21)$$

Thus for any given $\varepsilon > 0$ (and therefore any $\alpha < 3/4$) there exists a number N such that for any $n \geq N$ the canonical dimension $D < 0$. Hence there exists only a finite number of divergent diagrams. Rewriting (21) in the form

$$D \Big|_{\alpha = 3/4 - \varepsilon} = -2L\varepsilon + \frac{4 - E}{4}$$

It follows that $D \geq 0$ only if $E < 4$, i.e. $E = 0$ or $E = 2$ and the model is super-renormalizable.

Let us consider the case when $\alpha = 3/4$. Using (14) and (13) we have

$$D\Big|_{\alpha=3/4} = 1 - \frac{E}{4} \quad (22)$$

The equality (22) means that all divergent diagrams have only 0, 2, or 4 legs and the model is renormalizable.

Finally if $\alpha > 3/4$ we have

$$D\Big|_{\alpha>3/4} = \frac{L}{2} - n + 1 = \frac{2n - E + 1}{2} \geq \frac{1}{2} > 0 \quad (23)$$

Therefore if $\alpha > 3/4$ then all proper diagrams are divergent. \square

6 Examples

6.1 The stochastic process with $\alpha = 3/4$ and φ_4^4 theory

In this section the case $\alpha = 3/4$ is discussed in more detail. The stochastic process with this α is very interesting because the structure of ultraviolet divergencies in this case is similar to that of the φ_4^4 theory. Indeed, all proper vacuum diagrams as well as 2- and 4-leg diagrams are divergent.

However let us note an important difference. For the two-loop self-energy diagram (Fig. 2) we have

$$\Sigma_{\varkappa}(p) = \int_{-\varkappa}^{\varkappa} dk \int_{-\varkappa}^{\varkappa} dq \frac{|k|^{3/2}|q|^{3/2}|k+q-p|^{3/2}}{(k^2+m^2)(q^2+m^2)((k+q-p)^2+m^2)}$$

The leading divergence of this integral as $\varkappa \rightarrow \infty$ is proportional to $\varkappa^{1/2}$. Therefore the renormalized Lagrangian should have a counter-term of the form

$$C\varkappa^{1/2} : x^{(3/4)}(t)^2 :$$

The fractional derivative $x^{(3/4)}$ appears in this expression due to a factor $|p|^\alpha$ that according to the Feynman rules discussed in Sect.4 corresponds to every external leg.

The thorough consideration of the stochastic process with $\alpha = 3/4$ requires a further work.

6.2 The stochastic process with $\alpha = 5/8$

The stochastic process with $\alpha = 5/8$ is the simplest stochastic process that has ultraviolet divergencies. From Theorem it follows that in this case only the vacuum diagram (Fig. 3) is divergent. The contribution of this diagram is proportional to

$$\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \int_{-\infty}^{\infty} dk_3 \frac{|k_1|^{5/4} |k_2|^{5/4} |k_3|^{5/4} |k_1 + k_2 + k_3|^{5/4}}{(k_1^2 + m^2)(k_2^2 + m^2)(k_3^2 + m^2)((k_1 + k_2 + k_3)^2 + m^2)}$$

6.3 The stochastic process with $\alpha = 2/3$ and φ_3^4 theory

The interacting stochastic process with $\alpha = 2/3$ is interesting because it is the simplest stochastic process which has non-vacuum ultraviolet divergencies. From Theorem it follows that in this case only diagrams shown on (Fig. 2,3,4) are divergent. The structure of divergencies is similar to the φ_3^4 theory [16], however there is a difference. Besides the divergent constants corresponding to the vacuum diagrams the renormalized Lagrangian should also contain the term

$$C \ln \infty : x^{(2/3)}(t)^2 :$$

Here the divergence comes from the two-loop integral

$$\Sigma_{\infty}(p) = \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dq \frac{|k|^{4/3} |q|^{4/3} |k + q - p|^{4/3}}{(k^2 + m^2)(q^2 + m^2)((k + q - p)^2 + m^2)}$$

Renormalized correlation functions are

$$\langle x(t_1) \cdots x(t_N) e^{-\lambda U_{\infty}} \rangle / \langle e^{-\lambda U_{\infty}} \rangle$$

where

$$U_{\infty} = \int : x_{\infty}^{(2/3)}(\tau)^4 : g(\tau) d\tau + C \lambda \ln \infty \int : x_{\infty}^{(2/3)}(\tau)^2 : g(\tau) d\tau$$

We do not write the contribution from the vacuum diagrams.

7 Discussions and Conclusions

In this paper we have considered renormalization of the interacting stochastic process that has ultraviolet divergencies similar to those that appear

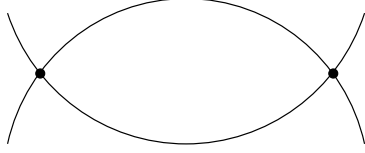


Fig. 1:

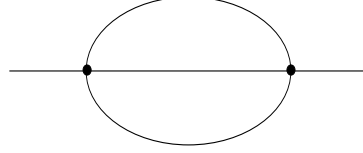


Fig. 2:

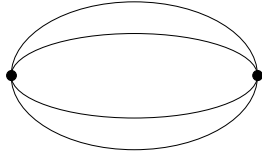


Fig. 3:

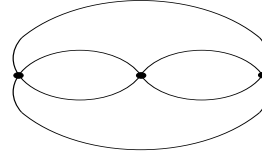


Fig. 4:

in multidimensional models of quantum field theory. We have studied the stochastic process in perturbation theory. It would be interesting to establish the existence of the stochastic process for $\alpha = 2/3$ nonperturbatively by using methods used in φ_3^4 theory [16]. Then properties of trajectories could be studied.

Especially interesting is the case when $\alpha = 3/4$ which is similar to the φ_4^4 theory. In this case the proof of the renormalizability of the stochastic process even in perturbation theory requires a further studying. We have discussed the renormalization in this case only at two-loop level. It would be also very interesting to investigate the critical behavior of the interacting stochastic process in the strong coupling regime on the lattice.

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